VRSPFMF2025-12-18

Zamudil 15 minut; manjka o Cauchy-Schwarzovi neenakosti

Trditev: Naj X, Y neodvisni sl. spremenljivki, ki imata matematični upanji. Tedaj ima mat. upanje tudi $X\cdot Y$ in velja

$$E(X\cdot Y)=E(X)\cdot E(Y).$$

Dokaz: (samo v zveznem primeru)

nedokazana trditev od zadnjič

$$E(\underbrace{X\cdot Y}_{f(X,Y),\ f(x,y)=x\cdot y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot y\cdot p_{(X,Y)}(x,y)\,dx\,dy \overset{\text{neodvisnost}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot y\cdot p_X(x)\cdot p_Y(y)\,dx\,dy =$$

$$= \int_{-\infty}^{\infty} x\,p_X(x)\,dx \cdot \int_{-\infty}^{\infty} y\,p_Y(y)\,dy = E(X)\cdot E(Y)$$

Def: Če za X in Y velja, da $E(X\cdot Y)=E(X)\cdot E(Y)$, sta X in Y nekorelirani, sicer sta korelirani.

Trditev: Obrat prejšnje trditve ne velja. ZDB ne velja nekoreliranost $\Rightarrow$ neodvisnost.

Dokaz: Zgled: $U:\begin{pmatrix} 0 & \pi/2 & \pi \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}$

$X=\cos U:\begin{pmatrix} 1 & 0 & -1 \\ 1/3 & 1/3 & 1/3\end{pmatrix}$

$Y=\sin U:\begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2/3 & 1/3\end{pmatrix}$

$$\boxed{E(X)=0} \quad \boxed{E(Y)=1/3} \quad \boxed{E(X\cdot Y)=0}$$

$X\cdot Y=\cos U \sin U:\begin{pmatrix} 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3\end{pmatrix} = \begin{pmatrix} 0 \\ 1\end{pmatrix}$

X in Y sta nekorelirani, nista pa neodvisni, ker $P(X=-1)=\frac{1}{3}$, $P(Y=0)=\frac{2}{3}$, $P(X=-1, Y=0)=\frac{1}{3}$

$P(X=-1)\cdot P(Y=0) \neq \frac{1}{3}\cdot\frac{2}{3}$

$X \backslash Y$	0	1	porazdelitev X
-1	$1/3$	0	$1/3$
0	0	$1/3$	$1/3$
1	$1/3$	0	$1/3$
	$2/3$	$1/3$	

porazdelitev Y

D.N. $X=\begin{pmatrix} x_1 & x_2 \\ p_1 & p_2\end{pmatrix}$ $Y=\begin{pmatrix} y_1 & y_2 \\ q_1 & q_2\end{pmatrix}$

Dokaži, da če imata X in Y le po 2 možni vrednosti, velja nekoreliranost $\Leftrightarrow$ neodvisnost

[Disperzija/varianca, kovarianca in korelacijski koeficient]

Def: Naj $\exists\ E(X^2)$. Disperzija oz. varianca sl. sprem. X je

$$D(X)\equiv Var(X) := E((X-E(X))^2) = E(X^2-2XE(X)+(E(X))^2) =$$

$$= E(X^2)-2E(X)E(X)+(E(X))^2 = E(X^2)-(E(X))^2$$

Disperzija meri razpršenost sl. sp. X okoli $E(X)$. Lastnosti:

1.) $D(X) \geq 0$; $D(X) = 0 \Leftrightarrow X$ izrojena, t.j. $\exists x_0 \in \mathbb{R}: P(X = x_0) = 1$

2.) $D(a \cdot X) = a^2 \cdot D(X) \quad \forall a \in \mathbb{R}$

3.) Za $a \in \mathbb{R}$ velja $E((X-a)^2) \geq D(X)$; enačaj velja $\Leftrightarrow a = E(X)$

<u>Dokaz:</u> $E((X-a)^2) = E(X^2 - 2aX + a^2) = E(X^2) - 2aE(X) + a^2 =$

$= (a - E(X))^2 - (E(X))^2 + E(X^2) = (a - E(X))^2 + D(X) \geq D(X)$, od koder sledi trditev.

<u>Def.:</u> Standardna deviacija ali odklon sl. sp. X je $\sigma(X) = \sqrt{D(X)}$.

Zanjo velja $\sigma(a \cdot X) = |a| \cdot \sigma(X)$, $a \in \mathbb{R}$.

Pregled rezultatov $E(X)$ in $D(X)$:

1.) Enakomerna diskretna porazd. $X: \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \frac{1}{n} & \frac{1}{n} & & \frac{1}{n} \end{pmatrix}$

$E(X) = \frac{x_1 + \cdots + x_n}{n}$, $D(X) = \frac{x_1^2 + \cdots + x_n^2}{n} - \left(\frac{x_1 + \cdots + x_n}{n}\right)^2$

2.) Binomska: $Bin(n,p)$, $n \in \mathbb{N}$, $p \in (0,1)$:

$E(X) = np$, $D(X) = npq$, $\sigma(X) = \sqrt{npq}$

3.) $Poi(\lambda)$, $\lambda > 0$: $E(X) = \lambda$, $D(X) = \lambda$, $\sigma(X) = \sqrt{\lambda}$

4.) $Geo(p)$ $\quad E(X) = \frac{1}{p}$ $\quad D(X) = \frac{q}{p^2}$

5.) $Pas(m,p)$ $\quad m \in \mathbb{N}$ $\quad p \in (0,1)$ $\quad E(X) = \frac{m}{p}$ $\quad D(X) = \frac{mq}{p^2}$

6.) Enakom. zv. por. na $[a,b]$: $E(X) = \frac{a+b}{2}$

7.) $N(\mu, \sigma)$ $\quad E(X) = \mu$, $D(X) = \sigma^2$, $\sigma(X) = \sigma$

8.) $\Gamma(b,c)$ $\quad E(X) = \frac{b}{c}$ $\quad D(X) = \frac{b}{c^2}$

9.) $\chi^2(n) = \Gamma(\frac{n}{2}, \frac{1}{2})$ $\quad E(X) = n$, $D(X) = 2n$

10.) $Exp(\lambda) = \Gamma(1, \lambda)$ $\quad E(X) = \frac{1}{\lambda}$ $\quad D(X) = \frac{1}{\lambda^2}$

<u>Dokažimo 4.:</u> $p_k = P(X = k) = q^{k-1}p$

$E(X) = \sum_{k=1}^{\infty} kpq^{k-1} = p \left(\sum_{k=0}^{\infty} q^k\right)' = p \left(\frac{1}{1-q}\right)' = p \cdot \frac{1}{(1-q)^2}$

$$\ldots = \frac{2}{p^2}$$

$$D(X) = \ldots = \frac{q}{p^2}$$

Zgled: Mečemo kocko. X je št. potrebnih metov, da pade šestica. Potem $X \sim geo(\frac{1}{6})$ in $E(X) = 6$, $D(X) = 30$ $\sigma(X) = 5{,}5$ $p_t = \left(\frac{5}{6}\right)^{t-1} \cdot \frac{1}{6}$.

Def: Kovarianca sl. sp. X in Y je št.

$$K(X,Y) = Cov(X,Y) := E\left((X - E(X)) \cdot (Y - E(Y))\right) =$$

$$= E(X \cdot Y - E(X) \cdot Y - X E(Y) + E(X) E(Y)) =$$

$$= E(X \cdot Y) - E(X) E(Y) \cancel{- E(Y) E(X)} \cancel{+ E(X) E(Y)} =$$

$$= E(X \cdot Y) - E(X) \cdot E(Y)$$

Lastnosti:

1.) $K(X,X) = D(X)$

2.) $K(X,Y) = 0 \iff X$ in Y nekorelirani

Posebej: X, Y neodvisni $\Rightarrow K(X,Y) = 0$

3.) K je simetrična in bilinearna (lin. v obeh arg.)

$K(Y,X) = K(X,Y)$

$K(aX + bY, Z) = aK(X,Z) + bK(Y,Z)$

$a, b \in \mathbb{R}$

4.) $|K(X,Y)| \leq \sqrt{D(X) D(Y)}$